Lecture 2A: Graph Theory I

UC Berkeley EECS 70
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Announcements!

- Read the Weekly Post
- Tarang's OH 4-6p in Woz Lounge (Zoom also–same link as lecture)
  - First 30 minutes for conceptual question
  - Last 90 minutes for reading Note 5 together and question about the note
  - Will not prioritize HW questions. Use regular OH for that.
- **HW 2** and **Vitamin 2** have been released, due **Thu** (grace period Fri)
- We are adding a bit more OH support, but also work on the HW early
- Throughout this lecture <u>**definitions**</u> will be underlined

Undirected Simple Graph Definitions

An undirected simple **graph** G = (V, *E*) is defined by

1. A set V of **vertices**. Sometimes we may call it a **node**.
2. A set *E* of **edges**

Where edges in *E* are of the form $\{u, v\}$ for u, v *in* V and $u \neq v$.

A graph being **simple** here means no parallel edges

A graph being **undirected** means there's no direction to the edges

Examples:

Directed Graph Definitions

Edges in a **directed graph** are defined as (u, v). That is, the order of the vertices matters. Therefore, $(u, v) \neq (v, u)$.

Examples:

Edge and Degree Definitions

Given an edge $e = \{u, v\}$ we say

- e is **incident** to u and v
- u and v *are* **neighbors**
- u and v *are* **adjacent**
- The **degree** of a vertex v is the number of incident edges
  - $deg(v) = |\{v \text{ in } V \mid \{u, v\} \text{ in } E\}|$

Examples:

Summary Questions I

How many nodes in this graph? ______

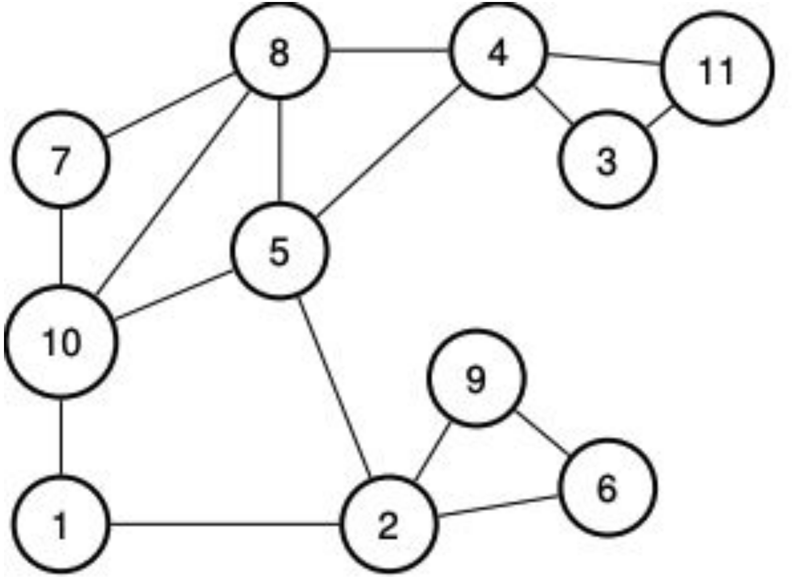
How many edges? ______

Which vertex has the max degree? ______

Which vertex has the min degree? ______

Which vertices is this edge incident on? _____

What is the sum of the degrees? _______



Handshake Lemma

Lemma: The sum of the degree of all the vertices is equal to $2|E|$

Proof:

Path, Cycles, Walks and Tours

Deals with Vertices (though may imply things about edges):

Path: A sequence of vertices in G, generally with no repeated vertices.

Cycle: A path in G where the only repeated vertex is the first one and last one.

Deals with Edges (though may imply things about vertices):

Walk: Is a sequence of edges with possible repeated vertex or edges.

Tour: A walk that starts and ends at the same vertex.

Eulerian walk: A walk where each edge is visited exactly once.

Eulerian tour: An Eulerian walk that starts and ends at the same vertex

Summary Questions II

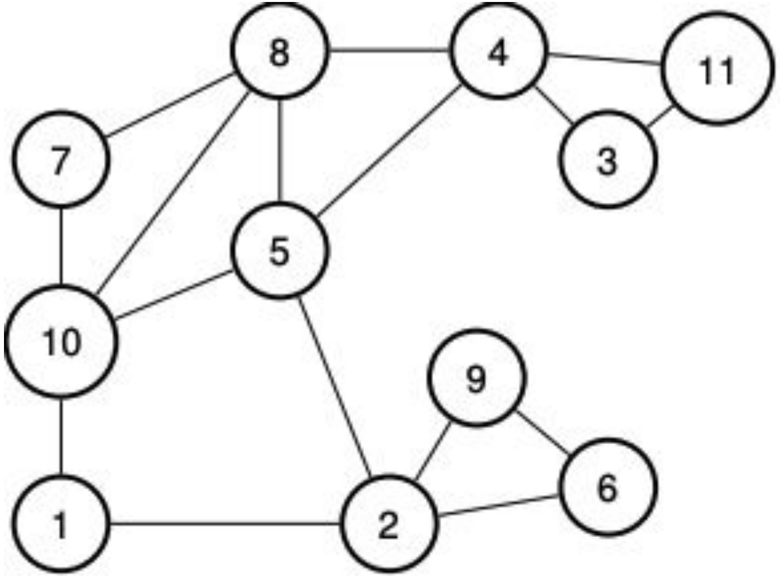
Give an example of length 3 cycle? ___________

Give an example of a path from 2 to 8? __________

Give the longest simple path? ________________

How many connected components are there? ____

Give an example of length 4 tour? ______________



Connectivity

A graph G is said to be **connected** if there exists a path between any two vertices.

Examples:

Any graph always consists of a collections of **connected components.** A connected component is a set of vertices in the graph that are connected.

Eulerian Tours

Eulerian walk: A walk where each edge is visited exactly once.

Eulerian tour: An Eulerian walk that starts and ends at the same vertex

Theorem: A undirected graph G has an Eulerian tour iff G is even degree, and connected.

Proof: *in the notes*

Summary Questions III

Is there an Eulerian Tour and if so provide a tour?

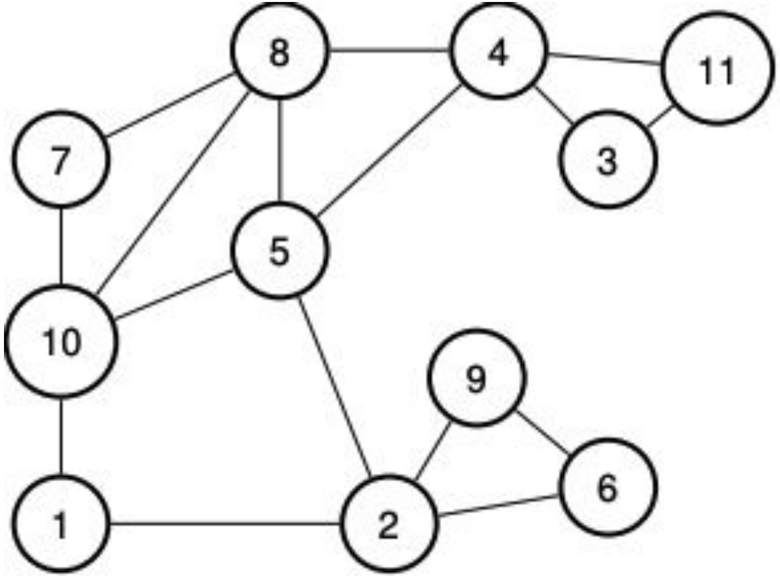
____

Why? ____________________

How many connected components now? _____

Connected components now? _____

What about now? _____



Graph Proof

False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

Proof: We use induction on the number of vertices $n \geq 1$

Base Case: There is only one graph with a single vertex and it has degree 0. Thus, vacuously true.

Inductive Hypothesis: Assume the claim is true for some $n \geq 1$

Inductive Step: We prove the claim is also true for $n + 1$. Consider an undirected graph with n vertices and each has degree greater than 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph with $(n + 1)$ vertices.

Since, the previous graph was connected, and x is connected to some node y then there's a path between x and any other vertex through y, since by definition there's a path from y to any other vertex. Thus, the graph is connected.

Minimum Edges for Connectivity

Theorem: Any connected graph with n vertices must have at least n-1 edges

Complete Graphs

A graph G is **complete** if it contains the maximum number of edges possible.

Examples:

Trees

The following definitions are all equivalent to show that a graph G is a **tree**.

1. G is connected and contains no cycles
2. G is connected and has n-1 edges (where $n = |V|$)
3. G is connected, and the remove of any single edge disconnects G
4. G has no cycles, and the addition of any single edge creates a cycle

Tree Definitions are Equivalent

Theorem: For a connected graph G it contains no cycles iff it has n-1 edges.
Proof:

Tree Definitions are Equivalent (cont.)

Theorem: For a connected graph G it contains no cycles iff it has n-1 edges.

Bipartite Graphs

A graph G is **bipartite** if the vertices can be split in two groups (L or R) and edges only go between groups.

Examples: