Counting

An introduction to combinatorics

Michael Psenka

Given ___, how many ___ are there?

100 vs 100!

- $100! \approx 10^{158}$
- Atoms in the observable universe:
  - $\approx 10^{80}$

Basis of counting: set cardinality

Definition. *The cardinality of a set S, denoted $|S|$, is given by the unique integer n such that:*

$$\textit{There exists a bijective map } f\colon S \leftrightarrow \{1, \dots, n\}.$$

(Fancy way of saying: just count up the elements)

- Shorthand: $[n] \coloneqq \{1, \dots, n\}$.

Fundamentals: cardinality rules

Let $A, B \subset S$. *Then the following cardinality rules hold:*

1. *Addition: If* $A \cap B = \emptyset$, *then* $|A \cup B| = |A| + |B|$.
   - **Proof.** Let $n := |A|, m := |B|$. We can construct $A \leftrightarrow [n]$ and $B \leftrightarrow \{n + 1, \ldots, n + m\}$, and further $A \cup B \leftrightarrow \{1, \ldots, n + m\}$.
2. *Subtraction: If* $B \subset A$, *then* $|A - B| = |A| - |B|$.
   - **Proof.** $A = (A - B) \cup B$ is a disjoint union, thus $|A| = |A - B| + |B|$.

Generalized addition: inclusion/exclusion

For general $A, B \subset S$, *we have that* $|A \cup B| = |A| + |B| - |A \cap B|$.

- **Proof.** Note that $A \cup \big(B - (A \cap B)\big)$ is a disjoint union by construction, and that $A \cup \big(B - (A \cap B)\big) = A \cup B$. Since $A \cap B \subset B$, we conclude that $|A \cup B| = |A| + |B - (A \cap B)| = |A| + |B| - |A \cap B|$.

Set multiplication: *outer product*

Definition. *Let A, B be sets. The outer product $A \times B$ is the set of ordered pairs (a, b) for all $a \in A, b \in B$.*

We then get the following product rule for set cardinalities:

$$|A \times B| = |A| \cdot |B|.$$

- **Proof** (sketch) Dictionary ordering. Enumerate A and B, then count out all $b \in B$ for the first $a \in A$, then for the 2^{nd} $a \in A$, etc...

"Set division": the quotient set

Definition. *Let A be a set, and $\sim$ an equivalence relation over A. The set A modulo $\sim$, denoted $A/{\sim}$, is the set of equivalence classes of A with respect to $\sim$.*

Example. Define $A \coloneqq \{1, \ldots, 10\}$, and the equivalence relation $a \sim b \coloneqq a \equiv b \pmod 2$. Then $A/{\sim} = \{\{1,3,5,7,9\},\{2,4,6,8,10\}\}$, and $|A/{\sim}| = 2$.

Quotient sets cardinality rule

Let A be a set and $\sim$ an equivalence relation over A. If every equivalence class has the same cardinality, denoted $|\sim|$, then the following holds:

$$|A/\sim| = \frac{|A|}{|\sim|}.$$

Proof. (Sketch) Note that we can write elements $a \in A$ as an outer product $a = (e, f)$, where e denotes the equivalence class of a and f denotes the membership of a within e. In this fashion, we can construct $A \leftrightarrow (A/\sim) \times (\sim)$, and conclude $|A| = |A/\sim||\sim|$.

Basic cardinality rules

1. *Addition: If* $A \cap B = \emptyset$, *then* $|A \cup B| = |A| + |B|$.
   - *General addition:* $|A \cup B| = |A| + |B| - |A \cap B|$.
2. *Subtraction: If* $B \subset A$, *then* $|A/B| = |A| - |B|$.

   $|A - B|$
3. *Multiplication:* $|A \times B| = |A||B|$.
4. *Division: If* $\sim$ *divides* A *evenly, then* $|A/\sim| = \frac{|A|}{|\sim|}$.

Principle of inclusion-exclusion

① $|A_1| + |A_2| - |A_1 \cap A_2|$

$\binom{n}{k}$

Multi-addition:

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{S \subset \{1,\ldots,n\} : |S| = k} |\cap_{i \in S} A_i|$$

$A \subset S$

$a \in A_1 \cup A_2 \cup \ldots \cup A_n$

$|A| = \sum_{s \in S} \mathbb{1}(s \in A)$

"indicator function"

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} = 1$$

② $|A_1| + |A_2| - |A_1 \cap A_2|$

Counting sequences

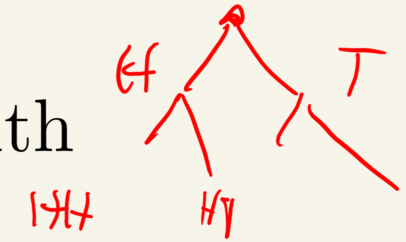
Sequences with replacement

Example: how many possible outcomes from flipping a coin 3 times?

Let A be a set of items to choose from. The space of k-long sequences of elements from A can be represented by the outer product space $A \times \cdots \times A$ (k copies).

(a, b, c)

- There are then $|A|^k$ sequences of k choices from A (with replacement).



coin flip 3 times → 2^3

Sequences without replacement

Example: how many possible 5 card hands from a standard deck?

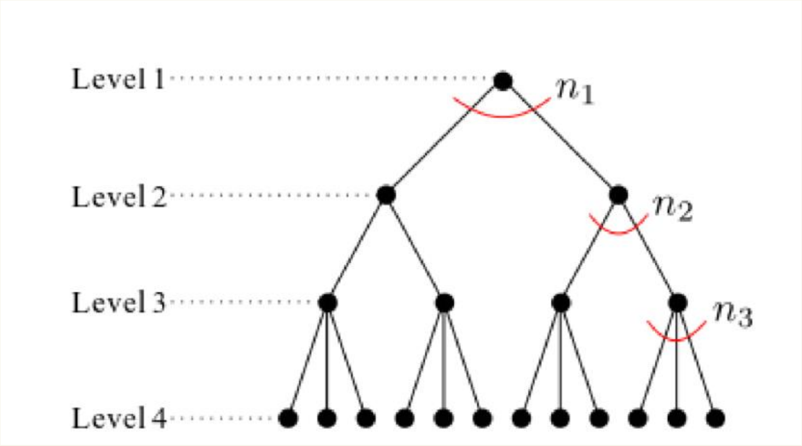
- Need a different model, since e.g. $(1,1,\ldots,1)$ no longer a valid choice.

Solution: model ordered pairs as choices from remaining altered set.
For example, $(1,1,1,1)$ corresponds to the cards (A, 2, 3, 4).

- Yields an outer product space $[n] \times [n-1] \times \cdots \times [n-(k-1)]$, which has $n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$ elements.

"First rule of counting"

If there are n_i choices to make at step i, the total number of ways to make a sequence of k choices is $n_1 \cdot n_2 \cdots n_k$.



Counting orderings

Formalizing an ordering

Definition. *Let A be a set, where $n := |A|$. An ordering of the set A is a bijective map $f: A \to \{1, \dots n\}$. The order of an element a is then given by $f(a)$.*

Central question: for a set A, how many orderings are there?

- **Note:** we can sufficiently count the orderings of the sets $\{1, \dots, n\}$.

Number of permutations.

Theorem. *The number of orderings of the set* $\{1, \dots, n\}$ *is* $n!$.

Proof intuition. Note that for any permutation, the element $\mathbf{1}$ has to be sent somewhere. For each position $\mathbf{1}$ is sent to, every permutation of the remaining $n - 1$ elements is a new permutation. As there are n positions to place $\mathbf{1}$, we then get $n(n-1)!$ total permutations.

$\{1, 2, 3, 4\}$

1, *, *, * — $3!$

*, 1, *, * — $3!$

$3! + 3! + 3! + 3!$

$= 4 \cdot 3! = 4!$

Number of permutations.

Theorem. *The number of orderings of the set* $\{1, \dots, n\}$ *is* $n!$.

Proof. We proceed via induction. Denote P_n the set of orderings of $\{1, \dots, n\}$. There exists only one bijective map between $\{1\}$ and itself, so $|P_1| = 1$ and the base case holds.

Assume $\{1, \dots, n-1\}$ has $(n-1)!$ orderings. Each $f : \{1, \dots, n\} \to \{1, \dots, n\}$ can be split into $f = (f(1), (f(2), \dots, f(n)))$, thus $P_n \leftrightarrow [n] \times P_{n-1}$, and $|P_n| = n \cdot |P_{n-1}| = n(n-1)! = n!$.

Deriving the combinations formula

Question: from a set of n items, how many ways to choose k of them?

- More formally: how many subsets of size k?

Idea: model a choice as the first k elements of an ordering.

- E.g. $\{5,1,2 \mid 4,3\}$ represents the choice $\{1,2,5\}$ from $\{1, \ldots, 5\}$.
  - Note the ordering of the choices does not matter.

Deriving the combinations formula

We can represent the space of size-k choices from a set of n elements as the following quotient space:

$$\frac{P_n}{\sim_1 \times \sim_2},$$

1. P_n: permutations of n elements.
2. $\sim_1$: permutations of the first k elements.
   - (order of choice doesn't matter)
3. $\sim_2$: permutations of the last $\mathrm{n}-k$ elements.
   - (order of elements we don't choose doesn't matter)

Deriving the combinations formula

$$\frac{P_n}{\sim_1 \times \sim_2},$$

1. P_n: permutations of n elements. $|P_n| = n!$
2. $\sim_1$: permutations of the first k elements. $|\sim_1| = k!$
3. $\sim_2$: permutations of the last $\mathrm{n} - k$ elements. $|\sim_2| = (n-k)!$

$$\left|\frac{P_n}{\sim_1 \times \sim_2}\right| = \frac{|P_n|}{|\sim_1| \cdot |\sim_2|} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

Choices when order matters

Only difference is the equivalence relation $\sim_1$ no longer holds, so we just get the following:

$$|\frac{P_n}{\sim_2}| = \frac{|P_n|}{|\sim_2|} = \frac{n!}{(n-k)!}$$

Other combinations examples

Example. Supposed we're tasked with counting the number of ways to order the letters in the word "Mississippi". There are 11 letters, yielding $11!$ orderings. However, permuting the "i's", "s's", or "p's" yield the same word. This generates 3 independent equivalence relations, which we can outer product together into a single equivalence relation:

$$\left|\frac{P_{11}}{\sim_i \times \sim_s \times \sim_p}\right| = \frac{|P_{11}|}{|\sim_i| \cdot |\sim_s| \cdot |\sim_p|} = \frac{11!}{4!4!2!}.$$

Other combinations examples

General practice on how to find the proper representation $A/\sim$ for a "given __, how many __?" problem:

1. $\boldsymbol{A}$: *How am I representing a choice?*
2. $\sim$: *Which representations correspond to the same choice?*

Combinatoric proof examples

Choosing fruits

Suppose we have a bin of infinite apples, oranges, and bananas. How many ways can we choose 5 fruits?

Solution 1. We can represent as a 5-tuple $(a_1, a_2, a_3, a_4, a_5)$, where each $a_i \in \{1,2,3\}$. This set has 3^5 elements. Since order of the fruits chosen doesn't matter, we have an equivalence of permutations of the 5 elements, whose classes are of size $|\sim| = 5!$. Thus, the number of choices is given by:

$$\left|\frac{\boldsymbol{A}}{\sim}\right| = \frac{\mathbf{3^5}}{\mathbf{5!}}.$$

← note, cannot be an integer!

(*Issue:* we've overcounted $| \sim |$, which isn't the same size everywhere)

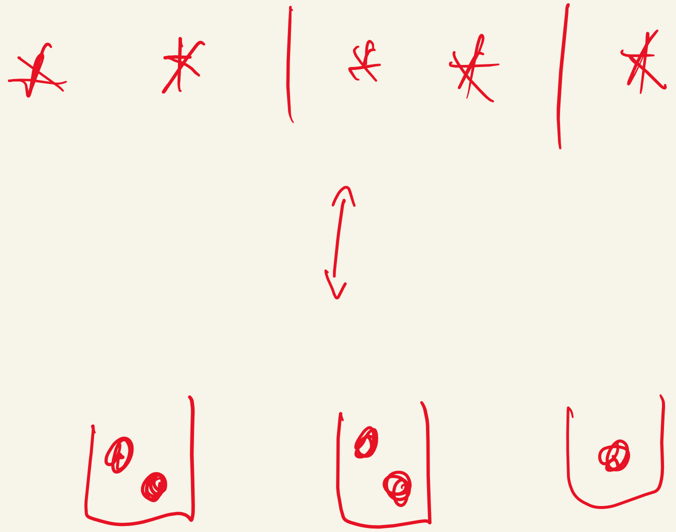
Stars and bars

How many ways are there to order a collection of $k-1$ *bars and* n *stars?*

Solution. $n+k\ -1$ total items, equivalence of permutations of the n stars and $k-1$ bars, yielding the following cardinality:

$$\frac{(n+k\ -1)!}{n!\,(k-1)!} = \binom{n+k-1}{k-1} = \binom{n+k-1}{n}.$$

Stars and bars



How many ways to put n balls in k bins?

Solution. “Balls”=stars and “bins” = (space between bars). Bijective relation to ordering of n stars between $k-1$ bars. Thus, the number of ways is $\binom{n+k-1}{n}$.

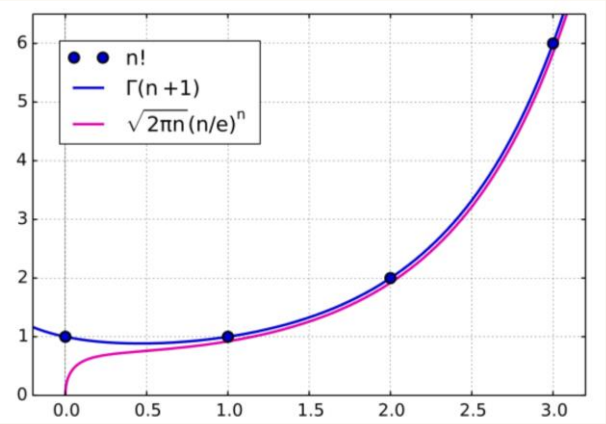
Stars and bars

How many ways to choose 5 fruits from 3 choices (apple, orange, banana)?

Solution. Bijective relation to balls and bins, where “bins” correspond to the type of fruit (apple, orange, banana), and “balls” are the 5 fruits chosen. Thus, the number of choices is $\binom{5+2}{5}$ **= 21.**

Fun example: sorting algorithms

- Algorithmic lower bound for sorting: $O(n\log(n))$
- Binary decision tree, must have at least $n!$ leaves.
  - $2^k \geq n!,$ thus at least $\log(n!)$ operations.
- Is $O(\log(n!))$ better than $O(n\log(n))$?
- Can use the *gamma function* $\Gamma(x)$ to derive *Stirling's approximation:*
  - $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$



- Conclusion: $O(n\log(n)) = O(\log(n!))$ is the best a sorting algorithm can possibly do.