Counting review

Countability

To infinity and beyond

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Intro question

- As many even integers as odd integers?

- As many even integers as integers?

Countably infinite sets

Definition. *The set S is said to be countable (countably infinite) if there exists a bijective map $f\colon S \leftrightarrow \mathbb{N}$.*

- In this sense, we can say that S and $\mathbb{N}$ have the same cardinality.

What sets are countable?

The smallest infinity

Theorem. *Every infinite subset of a countable set is countable.*

Building upwards

- $\mathbb{Z}$ is countable.

Building upwards

- $\mathbb{Z} \times \mathbb{Z}$ is countable.

Building upwards

- **Corollary.** *The following sets are countable:*

1. *The rational numbers* $\mathbb{Q}$.

2. *The sets* $\mathbb{Z}^{\times k} \coloneqq \mathbb{Z} \times \cdots \times \mathbb{Z}$ *(k copies).*

Building upwards

Theorem. *Any countable union of countable sets is countable.*

Another question

- Denote $\mathbb{Z}^{\mathbb{N}}$ as the set of (countably) infinite sequences of integers. Does there exist a bijection between the following:

$$\mathbb{Z}^{\mathbb{N}} \leftrightarrow \bigcup_{k=1}^{\infty} \mathbb{Z}^{\times k}?$$

The ceiling of countability

- The set $\{0,1\}^\mathbb{N}$ is not countable (uncountable).

Uncountable sets

- **Corollary.** *The following sets are uncountable:*

1. *The real numbers* $\mathbb{R}$.

2. *The set of subsets of* $\mathbb{N}$ *(denoted* $\mathcal{P}(\mathbb{N})$*).*

Uncountable(?) sets

The set of finite subsets of $\mathbb{N}$

Uncountable sets

Any nonempty closed interval $[a, b] \subset \mathbb{R}$ *is uncountable.*

Question: "how to measure size of uncountable sets"?

Measure zero and countability

Measure theory: measuring the size of (almost) arbitrary sets.

The Cantor set

The Cantor set $\cap_{k=1}^{\infty} C_k$ *is both measure zero and uncountable.*