Counting review

$n \quad (k-1)$

$\frac{(n+(k-1))!}{n!(k-1)!}$

$n \quad k$

$* \mid * \; * \; * \mid *$

$\frac{7!}{5!2!}$

$5 \quad 3$

$n! - |A_1 \cup A_2 \cup A_3 \cdots|$

$a \circ b$

$\sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{S \subset [n], |S|=k} |\cap_{i \in S} A_i| \right)$

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$(a+b)(a+b)\cdots(a+b)$

$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} = 1$

$a \in A_1 \cup \cdots A_n$

$\{1, 4, 7\}$

Countability

To infinity and beyond

Michael Psenka

Intro question

- As many even integers as odd integers? $E \quad O$

$f: E \to O \qquad n \to n+1 \qquad 4 + 1 = 5$

$7 - 1 = 6$

- As many even integers as integers?

$f: E \to \mathbb{Z} \qquad n \to \frac{n}{2} \qquad 9 \cdot 2 = 18$

$14 \to 7$

Countably infinite sets

Definition. *The set S is said to be countable (countably infinite) if there exists a bijective map f: $S \leftrightarrow \cancel{\mathbb{N}}$.* $\mathbb{Z}^+$ $\{1, 2, 3 \ldots\}$

- In this sense, we can say that S and $\mathbb{N}$ have the same cardinality.

$|S| = \infty$

What sets are countable?

$\mathbb{Z}^+$ id

$\mathbb{N} = \{0\} \cup \mathbb{Z}^+$ $f: \mathbb{N} \to \mathbb{Z}^+$ $f(n) = n + 1$

$\{-1\} \cup \{0\} \cup \mathbb{Z}^+ \longleftrightarrow \{-1\} \cup \mathbb{Z}^{-1}$

The smallest infinity

Theorem. *Every infinite subset of a countable set is countable.*

$S \subset A \xleftrightarrow{f} \mathbb{Z}^+$ $\underline{f} : \mathbb{Z}^+ \to S'$

$S' \subset \mathbb{Z}^+$ $a_1 \in S'$ $f(1) = a_1$ base case

$\{S' - a_1\}$

$f(n+1) = a_{n+1}$

$S' = f(S)$, $A' = f(A) = \mathbb{Z}^+$

ind. hyp. $S_{n+1} := \{S - a_1 - a_2 - \dots - a_n\}$ $a_{n+1} \in S_{n+1}$ ind. step

$a_1 = 2$

$f(1) = 2$ $f(2) = 2$ n, $f(n) = a_n$, "n^{th} lowest number"

Building upwards

- $\mathbb{Z}$ is countable.

$f : \mathbb{Z}^+ \to \mathbb{Z}$

$$\begin{array}{cccccc} 1, & 2, & 3, & 4, & 5, & 6, \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 1 & -1 & 2 & -2 & 3 \end{array}$$

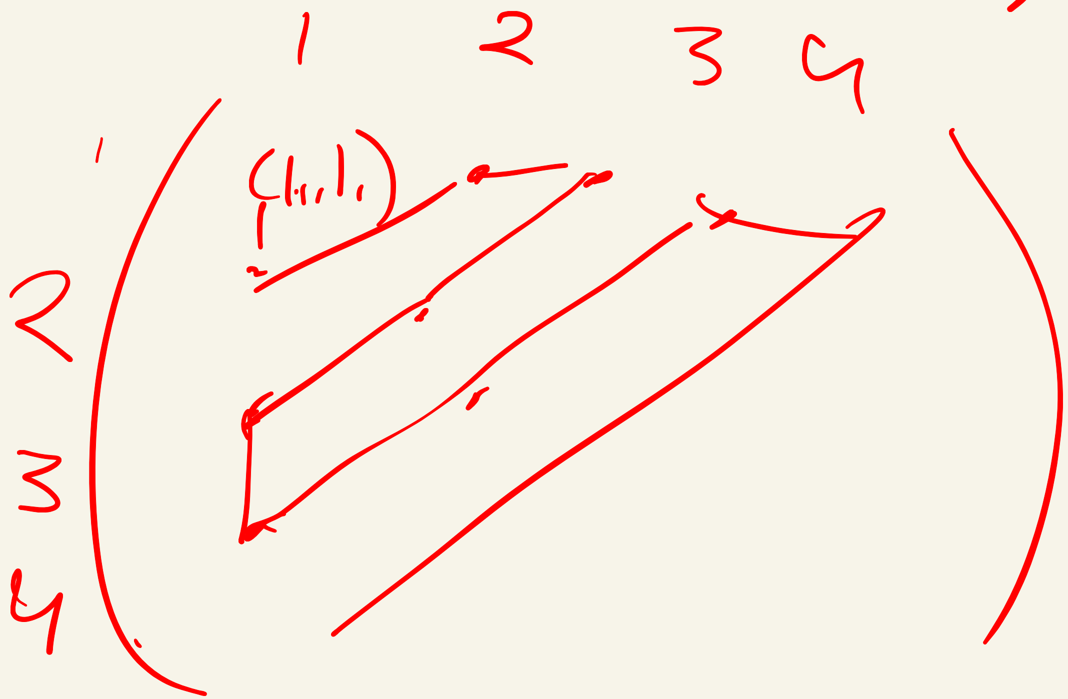
$$f(n) = \begin{cases} \frac{n}{2} & n \text{ is even} \\ -\frac{(n-1)}{2} & n \text{ is odd} \end{cases}$$

Building upwards

$\mathbb{Z}^+ \times \mathbb{Z}^+$

- $\mathbb{Z} \times \mathbb{Z}$ is countable.

(a, b)



$(1,1)$

$(2,1)$ $(1,2)$

$(3,1)$ $(2,2)$, $(1,3)$

$\mathbb{Z} \leftrightarrow \mathbb{Z}^+$

Building upwards

$(2,4)$
$(4,8)$

- **Corollary.** *The following sets are countable:*

1. *The rational numbers* $\mathbb{Q}$.

$\frac{p}{q} \longleftrightarrow S \subsetneq (p,q) \longleftrightarrow \mathbb{Z}^+$

$\Longrightarrow \mathbb{Q} \longleftrightarrow \mathbb{Z}^+$

2. *The sets* $\mathbb{Z}^{\times k} := \mathbb{Z} \times \cdots \times \mathbb{Z}$ *(k copies).*

$(\mathbb{Z} \times \mathbb{Z}) \times \mathbb{Z} \times \mathbb{Z}$

$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \qquad \mathbb{Z} \times \mathbb{Z} \qquad \mathbb{Z}$

Building upwards

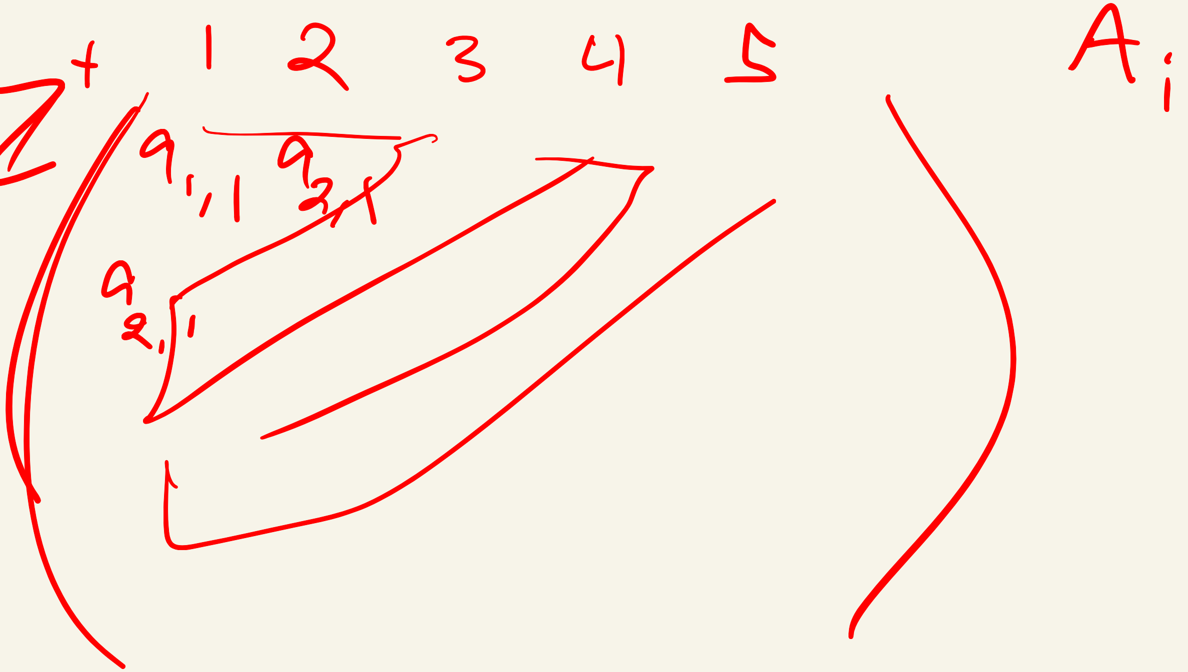
Theorem. *Any countable union of countable sets is countable.*

$$\bigcup_{i=1}^{\infty} A_i \subset \mathbb{Z}^+ \times \mathbb{Z}^+$$

(a, b)

which element in set b

which set



Another question

- Denote $\mathbb{Z}^{\mathbb{N}}$ as the set of (countably) infinite sequences of integers. Does there exist a bijection between the following:

$(2, 1, 0, 4, -1, -1, \ldots)$

$$\mathbb{Z}^{\mathbb{N}} \leftrightarrow \cup_{k=1}^{\infty} \mathbb{Z}^{\times k}?$$

k copies

$\mathbb{Z} \times \mathbb{Z} \times \ldots \mathbb{Z}$

$\mathbb{Z}^{\times \infty}$

$(1, 1, 1, 1, 1, \ldots) \notin \bigcup_{k=1}^{\infty} \mathbb{Z}^{\times k}$

$(4, 2, 1, 6)$

The ceiling of countability

- The set $\{0,1\}^\mathbb{N}$ is not countable (uncountable).

$(0, 1, 0, 0, \ldots)$

$\{0,1\}^\mathbb{N} \leftrightarrow \mathbb{Z}^+$

$$\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \vdots \end{array} \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 1 & 0 & 1 & \ldots \\ & & & & & & \end{pmatrix}$$

$s \in \{0,1\}^\mathbb{N}$

Need: (1) $s \in \{0,1\}^\mathbb{N}$

(2) $s \neq a_i \ \forall i \in \mathbb{Z}^+$

$s = 101\ldots$ (flip diagonal)

Uncountable sets

- **Corollary.** *The following sets are uncountable:*

1. *The real numbers* $\mathbb{R}$. $[0,1]$

$$r \leftrightarrow 0.1101\ldots$$

2. *The set of subsets of* $\mathbb{N}$ *(denoted* $\mathcal{P}(\mathbb{N})$*).*

$$\{1,4,7,10,\ldots\}$$

$$\updownarrow$$

$$(1,0,0,1,0,0,1,\ldots)$$

Uncountable(?) sets

The set of finite subsets of $\mathbb{N}$

$$S := \{A \subset \mathbb{N} : |A| < \infty\}$$

$$\mathcal{P}^k(\mathbb{N}) := \{A \subset \mathbb{N} : |A| = k\}$$

$$S = \bigcup_{k=0}^{\infty} \mathcal{P}^k(\mathbb{N}) \implies \text{countable}$$

(countable union of at most countable sets)

Uncountable sets

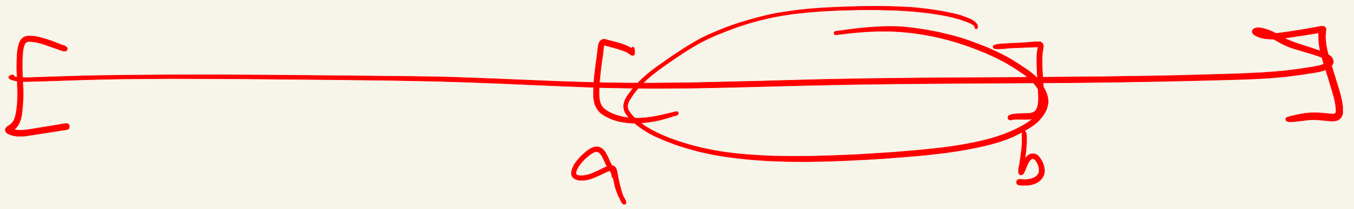
Any nonempty closed interval $[a,b] \subset \mathbb{R}$ *is uncountable.*

Question: "how to measure size of uncountable sets"?

Measure zero and countability

Measure theory: measuring the size of (almost) arbitrary sets.

$\mathbb{R}$



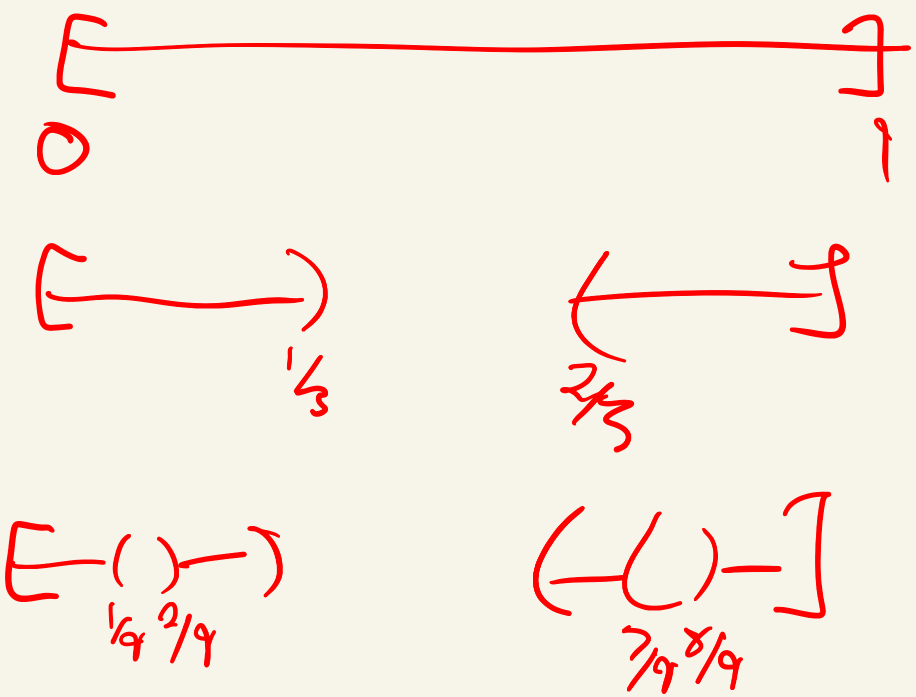
$b-a$

$A \subset \mathbb{R}$ $\qquad m(A) = 0$

$\{0\}$

The Cantor set

The Cantor set $\bigcap_{k=1}^{\infty} C_k$ *is both measure zero and uncountable.*



1

$2/3$

$4/9$

$\left(2/3\right)^n \to 0$

$0.020202\ldots$

$s = 101 \ldots$

$\times a_1$ 0 1 1 0 0 1 1 2 2 7

$\times a_2$ 1 1 1 1 1 1 1 1

$\times a_3$ 0 0 0 0 0 0 0

$s \in \bigcup^{\infty} \{0,1\}^{\times k}$

"i^{th} digit" of s = $\begin{cases} 1 & \text{if } i^{th} \text{ of } a_i = 0 \\ 0 & \text{otherwise} \end{cases}$

$\mathbb{Z}$

$\mathbb{Z}^{\mathbb{N}} \not\Leftrightarrow \bigcup_{k=1}^{\infty} \mathbb{Z}^{\times k}$

(blackboard from lecture)

$\mathbb{R}(0,1)$ uncountable

$\mathbb{R}[a,b]$

$0.1100101 \Leftrightarrow \{0,1\}^{\mathbb{N}}$

$x \to \frac{(x-a)}{b-a}$

$\mathbb{R}$ uncountable

1	2	3	4
1	1	0	1

$\bigcup^{\infty} \mathcal{P}_k(\mathbb{N})$

(set of all finite subsets)

$\mathcal{P}^{\varepsilon}(\mathbb{N}) = \bigcup^{\infty} \mathcal{P}^{k}(\mathbb{N})$

$\{1, 4, 7$

$(0.1001001$

(power set)

$\mathcal{P}(\mathbb{N}) \longleftrightarrow \{0,1\}^{\mathbb{N}}$

$1, 4, 7, 10$

$\mathcal{P}^{\varepsilon}(\mathbb{N}) =$ "all subsets of $\mathbb{N}$"

(blackboard from Q's after)

(every infinite subset of countable is countable proof)

$A \subset \mathbb{Z}^{+}$

$A\{3, 7, 9, 11, 13$

base ✓ $f(1) = a_1$ $a_1 \in A$

$f: \mathbb{Z}^{+} \to A$ $A_2 = A - \{a_1\}$ $f(2) = a_2$, $f(3) = a_3 \ldots$ $f(n) = a_n$

well-ordering

$A_{n+1} = A - \{a_1 \cup a_2 \cup \ldots a_n\}$

$a_{n+1} \in A_{n+1}$

$f(n+1) = a_{n+1}$

$a \in A$