Probability

Modeling uncertainty

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Random experiment (random variable)

Elements of probability theory

Sample space

Definition. *For a "random experiment", the sample space* Ω *is the set of possible outcomes of the random experiment.*

Events

Definition. *An event of a random variable with sample space* Ω *is a subset* $E \subset \Omega$.

Distribution

Definition. *A distribution* $\pi : \mathcal{P}(\Omega) \to \mathbb{R}$ *over a sample space* Ω *is a function of subsets of* Ω *that satisfies the following properties:*

1. *"Normalization":* $\pi(\emptyset) = 0, \pi(\Omega) = 1$.
2. *"Monotonicity":* $A \subset B \Rightarrow \pi(A) \leq \pi(B)$.
3. *"Additivity":* $A \cap B = \emptyset \Rightarrow \pi(A \cup B) = \pi(A) + \pi(B)$.

Measure-theoretic definition

1 Probability Spaces and Random Variables

Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space. The set Ω is called the *sample space*; its elements are called *outcomes*. The σ-algebra $\mathcal{H}$ may be called the grand *history*; its elements are called *events*. We repeat the properties of the probability measure $\mathbb{P}$; all sets here are events:

1.1 *Norming:*	$\mathbb{P}(\emptyset) = 0, \qquad \mathbb{P}(\Omega) = 1.$	
Monotonicity:	$H \subset K \quad \Rightarrow \quad \mathbb{P}(H) \le \mathbb{P}(K).$	
Finite additivity:	$H \cap K = \emptyset \quad \Rightarrow \quad \mathbb{P}(H \cup K) = \mathbb{P}(H) + \mathbb{P}(K).$	
Countable additivity:	(H_n) disjointed $\quad \Rightarrow \quad \mathbb{P}(\bigcup_n H_n) = \sum_n \mathbb{P}(H_n).$	
Sequential continuity:	$H_n \nearrow H \quad \Rightarrow \quad \mathbb{P}(H_n) \nearrow \mathbb{P}(H),$ $H_n \searrow H \quad \Rightarrow \quad \mathbb{P}(H_n) \searrow \mathbb{P}(H).$	
Boole's inequality:	$\mathbb{P}(\bigcup_n H_n) \quad \le \quad \sum_n \mathbb{P}(H_n).$	

Erhan Çinlar, *Probability and Stochastics* (you will not be tested on this)

Distribution (notes)

Definition. *A distribution* $\mathbb{P}: \mathcal{P}(\Omega) \to \mathbb{R}$ *over a sample space* Ω *is a function of subsets of* Ω, *first defined over the singletons* $\{a\}, a \in \Omega$ *such that:*

1. $\mathbb{P}(A) = \sum_{a \in A} \mathbb{P}(a)$,
2. $0 \leq \mathbb{P}(a) \leq 1$ *for all* $a \in \Omega$,
3. $\mathbb{P}(\Omega) = \sum_{a \in \Omega} \mathbb{P}(a) = 1$.

Equivalence of definitions

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Random variables

Definition. *A random variable is a double* $X = (\Omega, \pi)$, *where* Ω *is the sample space of* X*, and* π *is the distribution of* X.

Examples of random variables

Example: balls and bins

Example: “birthday paradox”

Example: Monty Hall problem