Continuous Random Variable I

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Continuous sample space

Probability mass function (PMF)

- For discrete random variable $\boldsymbol{X}$, probability mass function (PMF)denoted as $p_{\boldsymbol{X}(x)} = \mathbb{P}(\boldsymbol{X} = x)$ captures the probabilities of values that $\boldsymbol{X}$ can take.

- $\sum_x p_{\boldsymbol{X}(x)} = 1$

Probability density function (PDF)

- A random variable if called continuous if there is a nonnegative function $f_{\boldsymbol{X}}$ called probability density function (PDF) of $\boldsymbol{X}$ such that

$$\mathbb{P}(\boldsymbol{X} \in B) = \int_B f_{\boldsymbol{X}}(x)dx \quad \text{for every subset } B \subset \mathbb{R}.$$

- The probability that the value of $\boldsymbol{X}$ falls with in an interval is

$$\mathbb{P}(a \leq \boldsymbol{X} \leq b) = \int_b^a f_{\boldsymbol{X}}\,(x)dx$$

Probability density function (PDF)

$$\mathbb{P}(a \leq \boldsymbol{X} \leq b) = \int_{b}^{a} f_{\boldsymbol{X}}\,(x)dx$$

- The probability of $\boldsymbol{X}$ taking a single value is 0

- Normalization property

Example 1: Continuous uniform random variable

Spinning a wheel of fortune. The arrow continuously takes value between [0, 1]. Observe the number that the arrow points at.

Example 2: Piecewise constant PDF

Alice walks to class. It takes 15-20 min if it's sunny; it takes 20-25 min if it's rainy. Walking time being equally likely in each case. If in this city, the probability of a day is sunny is 2/3; a day is rainy is 1/3. What's the PDF of walking time $\boldsymbol{X}$

General piecewise constant PDF

Example 3: A PDF can take arbitrarily large value

Consider a random variable $\boldsymbol{X}$ with PDF

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \begin{cases} \dfrac{1}{a\sqrt{x}} & \text{if } 0 < x < 1 \\ 0 & otherwise \end{cases}$$

Summary of PDF

- A continuous random variable $\boldsymbol{X}$ with PDF $f_{\boldsymbol{X}}$

$$f_{\boldsymbol{X}}(x) \geq 0 \quad \forall x$$

$$\int_{-\infty}^{\infty} f_{\boldsymbol{X}}(x) dx = 1$$

For $B \subset \mathbb{R}$, $\mathbb{P}(\boldsymbol{X} \in B) = \int_B f_{\boldsymbol{X}}(x) dx$

Expectation

The expected value or expectation or mean of a continuous random variable $\boldsymbol{X}$ with PDF $f_{\boldsymbol{X}}$ is defined by

$$\mathbb{E}(\boldsymbol{X}) = \int_{-\infty}^{\infty} x\, f_{\boldsymbol{X}}(x) dx$$

Variance

The variance of a continuous random variable $\boldsymbol{X}$ with PDF $f_{\boldsymbol{X}}$ is defined by

$$Var(\boldsymbol{X}) = \mathbb{E}\left(\boldsymbol{X}^{\mathbf{2}}\right) - \mathbb{E}(\boldsymbol{X})^{\mathbf{2}}$$

$$= \int_{-\infty}^{\infty} x^2\, f_{\boldsymbol{X}}(x)dx - \left(\int_{-\infty}^{\infty} x\, f_{\boldsymbol{X}}(x)dx\right)^2$$

Example 4: mean and variance of the uniform random variable

Consider a uniform pdf over an interval [a, b]

Exponential Random Variable

- An exponential random variable has a PDF of the form

$$f_{\boldsymbol{X}}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Example 5.

Time till a small meteorite first lands anywhere in a desert is modeled as an exponential r.v. with a mean of 10 days. It is currently might night, what is the probability that a meteorite first lands between 6am to 6pm of the day?

Cumulative density function (CDF)

The CDF of a continuous random variable $\boldsymbol{X}$ with PDF $f_{\boldsymbol{X}}$ is denoted as $F_{\boldsymbol{X}}$

$\forall x,$

$$F_{\boldsymbol{X}(x)} = \mathbb{P}(\boldsymbol{X} \leq x) = \begin{cases} \sum_{k \leq x} p_{X(k)} & \text{if } \boldsymbol{X} \text{ is discrete} \\ \int_{-\infty}^{x} f_{\boldsymbol{X}}(t)dt & \text{if } \boldsymbol{X} \text{ is continuous} \end{cases}$$

Example 1

Example 2

Properties of a CDF $F_{\boldsymbol{X}(x)} = \mathbb{P}(\boldsymbol{X} \leq x)$