CS 70 Discrete Mathematics and Probability Theory

 $Summer \ 2022 \quad \hbox{Jingjia Chen, Michael Psenka and Tarang Srivastava}$

DIS 4B

1 Countability: True or False

- (a) The set of all irrational numbers $\mathbb{R}\setminus\mathbb{Q}$ (i.e. real numbers that are not rational) is uncountable.
- (b) The set of integers x that solve the equation $3x \equiv 2 \pmod{10}$ is countably infinite.
- (c) The set of real solutions for the equation x + y = 1 is countable.

For any two functions $f: Y \to Z$ and $g: X \to Y$, let their composition $f \circ g: X \to Z$ be given by $f \circ g = f(g(x))$ for all $x \in X$. Determine if the following statements are true or false.

- (d) f and g are injective (one-to-one) $\implies f \circ g$ is injective (one-to-one).
- (e) f is surjective (onto) $\implies f \circ g$ is surjective (onto).

2 Countability Practice

(a) Do (0,1) and $\mathbb{R}_+ = (0,\infty)$ have the same cardinality? If so, either give an explicit bijection (and prove that it is a bijection) or provide an injection from (0,1) to $(0,\infty)$ and an injection from $(0,\infty)$ to (0,1) (so that by Cantor-Bernstein theorem the two sets will have the same cardinality). If not, then prove that they have different cardinalities.



3 Unions and Intersections

For each of the following, decide if the expression is "Always Countable", "Always Uncountable", "Sometimes Countable, Sometimes Uncountable".

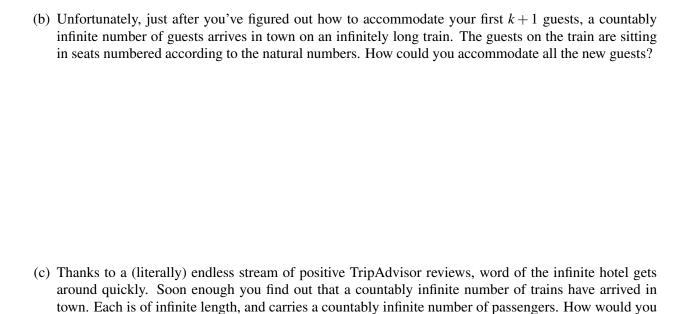
For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

- (a) $A \cap B$, where A is countable, and B is uncountable
- (b) $A \cup B$, where A is countable, and B is uncountable
- (c) $\bigcap_{i \in A} S_i$ where A is a countable set of indices and each S_i is an uncountable set.

4 Hilbert's Hotel

You don't have any summer plans, so you decide to spend a few months working for a magical hotel with a countably infinite number of rooms. The rooms are numbered according to the natural numbers, and all the rooms are currently occupied. Assume that guests don't mind being moved from their current room to a new one, so long as they can get to the new room in a finite amount of time (i.e. guests can't be moved into a room infinitely far from their current one).

(a) A new guest arrives at the hotel. All the current rooms are full, but your manager has told you never to turn away a guest. How could you accommodate the new guest by shuffling other guests around? What if you instead had k guest arrive, for some fixed, positive $k \in \mathbb{Z}$?



accommodate all the new passengers?