

1 Interesting Gaussians

(a) If $X \sim N(0, \sigma_X^2)$ and $Y \sim N(0, \sigma_Y^2)$ are independent, then what is $\mathbb{E}[(X + Y)^k]$ for any *odd* $k \in \mathbb{N}$?

(b) Let $f_{\mu, \sigma}(x)$ be the density of a $N(\mu, \sigma^2)$ random variable, and let X be distributed according to $\alpha f_{\mu_1, \sigma_1}(x) + (1 - \alpha) f_{\mu_2, \sigma_2}(x)$ for some $\alpha \in [0, 1]$. Compute $\mathbb{E}[X]$ and $\text{Var}(X)$. Is X normally distributed?

2 Continuous Joint Densities

The joint probability density function of two random variables X and Y is given by $f(x,y) = Cxy$ for $0 \leq x \leq 1, 0 \leq y \leq 2$, and 0 otherwise (for a constant C).

(a) Find the constant C that ensures that $f(x,y)$ is indeed a probability density function.

(b) Find $f_X(x)$, the marginal distribution of X .

(c) Find the conditional distribution of Y given $X = x$.

(d) Are X and Y independent?

3 Binomial Concentration

Here, we will prove that the binomial distribution is *concentrated* about its mean as the number of trials tends to ∞ . Suppose we have i.i.d. trials, each with a probability of success $1/2$. Let S_n be the number of successes in the first n trials (n is a positive integer), and define

$$Z_n := \frac{S_n - n/2}{\sqrt{n}/2}.$$

(a) What are the mean and variance of Z_n ?

(b) What is the distribution of Z_n as $n \rightarrow \infty$?

(c) Use the bound $\mathbb{P}[Z > z] \leq (\sqrt{2\pi}z)^{-1} e^{-z^2/2}$ when Z is a standard normal in order to approximately bound $\mathbb{P}[S_n/n > 1/2 + \delta]$, where $\delta > 0$.