Jingjia Chen, Michael Psenka and Tarang Srivastava

1 Coins of LLSE

There are 3 coins in a bag, with biases $1/3$, $1/2$, $2/3$ (bias means the chance the coin will be heads). After picking a coin, you flip the coin 4 times. Let X_i be the indicator variable that the ith flip is heads. Let $X = \sum_{1 \le i \le 2} X_i$ and $Y = \sum_{3 \le i \le 4} X_i$. Find $L(Y \mid X)$. Recall that

$$L(Y \mid X) = \mathbb{E}[Y] + \frac{\text{cov}(Y,X)}{\text{Var}(X)}(X - \mathbb{E}[X]).$$

2 LLSE

We have two bags of balls. The fractions of red balls and blue balls in bag A are $2/3$ and $1/3$ respectively. The fractions of red balls and blue balls in bag B are $1/2$ and $1/2$ respectively. Someone gives you one of the bags (unmarked) uniformly at random. You then draw 6 balls from that same bag with replacement. Let X_i be the indicator random variable that ball i is red. Now, let us define $X = \sum_{1 \le i \le 3} X_i$ and $Y = \sum_{4 \le i \le 6} X_i$.

(a) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(b) Compute $\text{Var}(X)$.

(c) Compute $\text{cov}(X,Y)$. (*Hint*: Recall that covariance is bilinear.)

(d) Now, we are going to try and predict Y from a value of X. Compute $L(Y \mid X)$, the best linear estimator of Y given X. (*Hint*: Recall that

$$L(Y \mid X) = \mathbb{E}[Y] + \frac{\text{cov}(X,Y)}{\text{Var}(X)}\big(X - \mathbb{E}[X]\big).$$

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3 Balls in Bins Estimation

We throw $n > 0$ balls into $m \geq 2$ bins. Let X and Y represent the number of balls that land in bin 1 and 2 respectively.

(a) Calculate $\mathbb{E}[Y \mid X]$. [*Hint*: Your intuition may be more useful than formal calculations.]

(b) What is $L[Y \mid X]$ (where $L[Y \mid X]$ is the best linear estimator of Y given X)? [*Hint*: Your justification should be no more than two or three sentences, no calculations necessary! Think carefully about the meaning of the conditional expectation.]

(c) Unfortunately, your friend is not convinced by your answer to the previous part. Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(d) Compute $\text{Var}(X)$.

(e) Compute $\text{cov}(X,Y)$.

(f) Compute $L[Y \mid X]$ using the formula. Ensure that your answer is the same as your answer to part (b).