

Due: Jun 30, 2022 11:59pm
Grace period until Jul 1, 2022 11:59pm

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Pebbles

Suppose you have a rectangular array of pebbles, where each pebble is either red or blue. Suppose that for every way of choosing one pebble from each column, there exists a red pebble among the chosen ones. Prove that there must exist an all-red column.

2 Contraposition

Prove the statement "if $x + y < z + w$, then $x < z$ or $y < w$ ".

- (a) Prove this statement with a direct proof.
- (b) Prove this statement via contraposition.

3 Fibonacci Proof

Let F_i be the i^{th} Fibonacci number, defined by $F_{i+2} = F_{i+1} + F_i$ and $F_0 = 0, F_1 = 1$. Prove that

$$\sum_{i=0}^n F_i^2 = F_n F_{n+1}.$$

4 Airport

Suppose that there are $2n + 1$ airports where n is a positive integer. The distances between any two airports are all different. For each airport, exactly one airplane departs from it and is destined for the closest airport. Prove by induction that there is an airport which has no airplanes destined for it.

5 Coloring Trees

Prove that all trees with at least 2 vertices are *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[Hint: Use induction on the number of vertices.]

6 Planarity

- (a) Prove that $K_{3,3}$ is nonplanar.
- (b) Consider graphs with the property T : For every three distinct vertices v_1, v_2, v_3 of graph G , there are at least two edges among them. Use a proof by contradiction to show that if G is a graph on ≥ 7 vertices, and G has property T , then G is nonplanar.

7 Bipartite Graphs

An undirected graph is bipartite if its vertices can be partitioned into two disjoint sets L, R such that each edge connects a vertex in L to a vertex in R (so there does not exist an edge that connects two vertices in L or two vertices in R).

- (a) Suppose that a graph G is bipartite, with L and R being a bipartite partition of the vertices. Prove that $\sum_{v \in L} \deg(v) = \sum_{v \in R} \deg(v)$.
- (b) Suppose that a graph G is bipartite, with L and R being a bipartite partition of the vertices. Let s and t denote the average degree of vertices in L and R respectively. Prove that $s/t = |R|/|L|$.
- (c) Prove that a graph is bipartite if and only if it can be 2-colored. (A graph can be 2-colored if every vertex can be assigned one of two colors such that no two adjacent vertices have the same color).

8 Build-Up Error?

What is wrong with the following "proof"? In addition to finding a counterexample, you should explain what is fundamentally wrong with this approach, and why it demonstrates the danger of build-up error.

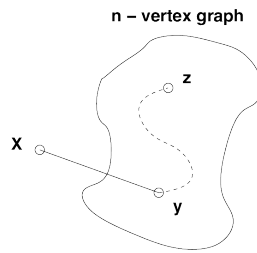
False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

Proof: We use induction on the number of vertices $n \geq 1$.

Base case: There is only one graph with a single vertex and it has degree 0. Therefore, the base case is vacuously true, since the if-part is false.

Inductive hypothesis: Assume the claim is true for some $n \geq 1$.

Inductive step: We prove the claim is also true for $n + 1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n + 1)$ vertices, as shown below.



All that remains is to check that there is a path from x to every other vertex z . Since x has degree at least 1, there is an edge from x to some other vertex; call it y . Thus, we can obtain a path from x to z by adjoining the edge $\{x, y\}$ to the path from y to z . This proves the claim for $n + 1$.