CS 70 Discrete Mathematics and Probability Theory
Summer 2022 Jingjia Chen, Michael Psenka and Tarang Srivastava HW 3

Due: Jun 7, 2022 11:59pm
Grace period until Jul 8, 2022 11:59pm

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 A Number Theoretic Warm-Up

Answer each of the following questions with brief justification.

(a) What is the last digit of 15^{2021}?

(b) What is the inverse of 3 modulo 20?

(c) For how many values of $a \pmod{10}$ does a^{-1} exist modulo 10?

2 CRT Decomposition

In this problem we will find $3^{302} \mod 385$.

(a) Write 385 as a product of prime numbers in the form $385 = p_1 \times p_2 \times p_3$.

(b) Use Fermat's Little Theorem to find $3^{302} \mod p_1$, $3^{302} \mod p_2$, and $3^{302} \mod p_3$.

(c) Let $x = 3^{302}$. Use part (b) to express the problem as a system of congruences (modular equations $\mod 385$). Solve the system using the Chinese Remainder Theorem. What is 3^{302} $\mod 385$?

3 Sparsity of Primes

A prime power is a number that can be written as p^i for some prime p and some positive integer i. So, $9 = 3^2$ is a prime power, and so is $8 = 2^3$. $42 = 2 \cdot 3 \cdot 7$ is not a prime power.

Prove that for any positive integer k, there exists k consecutive positive integers such that none of them are prime powers.

Hint: This is a Chinese Remainder Theorem problem. We want to find x such that $x+1, x+2, \ldots, x+k$ are all not powers of primes. We can enforce this by saying that $x+1$ through $x+k$ each must have two distinct prime divisors.

4 Baby Fermat

Assume that a does have a multiplicative inverse mod m. Let us prove that its multiplicative inverse can be written as $a^k \pmod{m}$ for some $k \geq 0$.

(a) Consider the infinite sequence $a, a^2, a^3, \ldots \pmod{m}$. Prove that this sequence has repetitions. (**Hint:** Consider the Pigeonhole Principle.)

(b) Assuming that $a^i \equiv a^j \pmod{m}$, where $i > j$, what is the value of $a^{i-j} \pmod{m}$?

(c) Prove that the multiplicative inverse can be written as $a^k \pmod{m}$. What is k in terms of i and j?

5 Euler's Totient Function

Euler's totient function is defined as follows:

$$\phi(n) = |\{i : 1 \leq i \leq n, \texttt{gcd}(n,i) = 1\}|$$

In other words, $\phi(n)$ is the total number of positive integers less than or equal to n which are relatively prime to it. We develop a general formula to compute $\phi(n)$.

(a) Let p be a prime number. What is $\phi(p)$?

(b) Let p be a prime number and k be some positive integer. What is $\phi(p^k)$?

(c) Show that if $\gcd(m,n) = 1$, then $\phi(mn) = \phi(m)\phi(n)$. (Hint: Use the Chinese Remainder Theorem.)

(d) Argue that if the prime factorization of $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then

$$\phi(n) = n \prod_{i=1}^{k} \frac{p_i - 1}{p_i}.$$

6 Using RSA

Kevin and Bob decide to apply the RSA cryptography so that Kevin can send a secret message to Bob.

1. Assuming $p = 3$, $q = 11$, and $e = 7$, what is d? Calculate the exact value.
2. Following part (a), what is the original message if Bob receives 4? Calculate the exact value.

7 Breaking RSA

Eve is not convinced she needs to factor $N = pq$ in order to break RSA. She argues: "All I need to know is $(p-1)(q-1)$... then I can find d as the inverse of e mod $(p-1)(q-1)$. This should be easier than factoring N." Prove Eve wrong, by showing that if she knows $(p-1)(q-1)$, she can easily factor N (thus showing finding $(p-1)(q-1)$ is at least as hard as factoring N).

8 RSA with Three Primes

Show how you can modify the RSA encryption method to work with three primes instead of two primes (i.e. $N = pqr$ where p, q, r are all prime), and prove the scheme you come up with works in the sense that $D(E(x)) \equiv x \pmod{N}$.